

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4078

A DISCUSSION OF CONE AND FLAT-PLATE
REYNOLDS NUMBERS FOR EQUAL RATIOS OF THE LAMINAR SHEAR
TO THE SHEAR CAUSED BY SMALL VELOCITY FLUCTUATIONS IN
A LAMINAR BOUNDARY LAYER

By Neal Tetervin

Langley Aeronautical Laboratory
Langley Field, Va.



Washington

August 1957

AFM 6
TECHNICAL NOTE
AFL 2811



0066949

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4078

A DISCUSSION OF CONE AND FLAT-PLATE
REYNOLDS NUMBERS FOR EQUAL RATIOS OF THE LAMINAR SHEAR
TO THE SHEAR CAUSED BY SMALL VELOCITY FLUCTUATIONS IN
A LAMINAR BOUNDARY LAYER

By Neal Tetervin

SUMMARY

By use of the linear theory of boundary-layer stability, an approximate relation is derived between the Reynolds number on a cone and the Reynolds number on a flat plate for equal closeness to transition; the assumption is made that the ratio of the laminar shear to the shear caused by the velocity fluctuations in the laminar boundary layer is an indication of the closeness to transition. The fluctuations on plate and cone are assumed to be of the same type and to be periodic in the direction of flow. By use of Schlichting's calculated amplification ratios for incompressible flow, the approximate relation is made specific. This specific relation is roughly that, at equal ratios of oscillation shear to laminar shear, the cone Reynolds number based on the distance from the apex exceeds the plate Reynolds number based on the distance from the leading edge by twice the minimum critical Reynolds number on the plate. This relation requires that the amplitude of the disturbance be equal on cone and plate where amplification begins. The frequency on the cone is the frequency that results in the maximum amplification at a Reynolds number; the frequency on the plate is the frequency that results in the maximum amplification at the corresponding Reynolds number on the plate and is in general not the same as the frequency on the cone.

Although an exact analysis of the transition problem is not given, nor is there given even an exact analysis of the stability of the laminar boundary layer on a cone, the indication is that the ratio of the cone Reynolds number for transition, based on the distance to the cone apex, to the plate Reynolds number for transition, based on distance to the leading edge, is not in general equal to 3, as has been suggested by other investigators. The analysis indicates that the ratio varies from 3 when transition occurs at the minimum critical Reynolds number to unity when transition occurs at a large multiple of the critical Reynolds number. An examination of two sets of data does not lead to a definite conclusion concerning the validity of the results obtained.

INTRODUCTION

The significant practical effects of the difference between the skin-friction and heat-transfer coefficients associated with laminar boundary layers and those associated with turbulent boundary layers make important the study of the transition from laminar to turbulent flow. At present the useful information concerning transition is almost entirely derived from experiment. There is, however, a theory of the stability of laminar flow for both incompressible and compressible flow (ref. 1) which, although not sufficiently developed to be of direct practical use, is often useful in a qualitative sense; that is, the theory can predict the direction of the effect on transition of changes in pressure gradient, heat transfer, Mach number, and Reynolds number if the initial disturbances in the laminar flow that eventually produce transition to turbulent flow are sufficiently small. Disturbances that are too large to be described by the linear equations are often introduced by local surface imperfections or by distributed roughness.

Although the theory of reference 1 gives the direction of the effect on transition of a change in pressure gradient, heat transfer, Mach number, and Reynolds number, it can in no case predict the Reynolds number for transition. This limitation exists even when the imposed disturbances are small enough to allow their initial behavior to be predicted by a linearized disturbance equation. The reason is that when the disturbances have grown large enough to cause transition, they have also become too large for their behavior to be described by the linear equations. The information on transition is therefore obtained from experiment, with the stability theory serving at most as a guide for expected effects.

A simple example of boundary-layer flow is the flow over a flat plate in the absence of a pressure gradient; this flow has been extensively investigated at both subsonic and supersonic speeds. At supersonic speeds the flow over a cone at zero angle of attack is another example of flow with no pressure gradient. Since many bodies have cone-like forebodies, the skin-friction and heat-transfer coefficients on cones are subjects of study. A simple relation is known to exist between the skin-friction and heat-transfer coefficients of cones and those of flat plates for either laminar flow (ref. 2) or turbulent flow (ref. 3). Because there is, however, no known relation between the transition position on a cone and that on a flat plate, this simple relation is not as useful as it would otherwise be.

In order to obtain some information concerning the relation between the transition positions on cones and those on plates, use is made of results obtained by Battin and Lin (ref. 4) concerning the relation between the amplification of disturbances on cones and plates. Their conclusion is that, for a disturbance of a given time frequency, the

amplification of the disturbance between two boundary-layer Reynolds numbers on a cone will be A^3 if the amplification between the same boundary-layer Reynolds numbers on a plate is A . This result is the starting point for the present analysis, the purpose of which is to investigate the possibility of obtaining a useful approximate relation between the Reynolds number on a cone and that on a plate for equal closeness to transition. The analysis is based on the linear stability theory and on the assumption that the ratio of the shear caused by the disturbance to the laminar shear is an indication of the closeness to transition. The information so obtained may be useful if the difference between the transition Reynolds number and the Reynolds numbers at which the relations for equal shear ratio become inapplicable is small.

SYMBOLS

A	amplification ratio
b	ratio of v' to u'
c	$c_r + ic_1$
c_r	wave velocity of a disturbance of a single frequency
c_1	amplification parameter; $c_1 < 0$ for a decaying disturbance; $c_1 = 0$ for a neutral disturbance; $c_1 > 0$ for a growing disturbance
C_1	boundary-layer Reynolds number parameter
$C_{f, \text{lam}}$	laminar shearing stress coefficient, $\frac{\tau_w}{\rho_\delta U_\delta^2}$ $\frac{\tau_w}{2}$
$F()$	function of $()$
k	correlation coefficient, $\frac{\overline{uv}}{u'v'}$
M	Mach number
r	perpendicular distance from point x, y to axis of symmetry of cone
S	amplification parameter
t	time

u	instantaneous value of velocity fluctuation in x-direction
u'	root-mean-square of velocity fluctuation in x-direction
U_δ	velocity at outer edge of boundary layer
U	velocity inside boundary layer and parallel to surface
v	instantaneous value of velocity fluctuation in y-direction
v'	root-mean-square of velocity fluctuation in y-direction
x	distance measured along surface from leading edge or apex
y	distance from surface measured perpendicular to surface
R_{δ^*}	boundary-layer Reynolds number based on displacement thickness, $\frac{U_\delta \delta^*}{\nu}$
R_x	Reynolds number based on distance x , $\frac{U_\delta x}{\nu}$
T	temperature
α	wave number of disturbance, $\frac{2\pi}{\lambda}$
δ	boundary-layer thickness
δ^*	displacement thickness of boundary layer, $\int_0^\infty \left(1 - \frac{\rho U}{\rho_\delta U_\delta}\right) dy$
λ	wavelength of disturbance
ν	kinematic viscosity at outer edge of boundary layer
ρ	density
τ_{turb}	shear caused by oscillations, $-\rho \bar{u'v'}$
τ_{lam}	laminar shear, $\mu \frac{dU}{dy}$
μ	viscosity
ϕ	amplitude function
—	mean value

Subscripts:

c	cone
0	at the beginning of amplification
min	minimum
max	maximum
p	plate
T	transition
x	at distance x
δ	at outer edge of boundary layer
w	wall

ANALYSIS

Battin and Lin (ref. 4) stated the result that, if a disturbance of a given time frequency grows A times between two Reynolds numbers $R_{\delta^*,1}$ and $R_{\delta^*,2}$ on a plate, then a disturbance of the same time frequency will grow A^3 times between $R_{\delta^*,1}$ and $R_{\delta^*,2}$ on a cone. This result can be obtained by noting that the amplification ratio as given in reference 5 is (in the present notation)

$$A = \frac{\left(\frac{u'}{U_\delta}\right)_x}{\left(\frac{u'}{U_\delta}\right)_0} = e^{\int_{t_0}^{t_x} \alpha c_1 dt} \quad (1)$$

where t_0 is the time at which the disturbance, which travels like a wave, first reaches the part of the boundary layer where it is amplified, and t_x is the time at which it arrives at x .

The use of equation (1) to calculate the amplification on a cone implies that the ratio of the local boundary-layer thickness to the local cone radius is vanishingly small. In this case the equations that describe the local stability characteristics of the boundary layer are the same as

those for a flat plate. The effect on the disturbance of the change in surface area of the cone with increase in distance downstream is, however, not available from the equations for a vanishingly thin boundary layer; consequently the use of equation (1) for a cone is an approximation. In the present analysis the assumption is also made that the disturbances on both plate and cone are of the type $\phi(y)e^{i\alpha(x-ct)}$.

In order to express the amplification ratio as a function of the boundary-layer Reynolds numbers at times t_0 and t_x , make use of the fact that a disturbance composed of a single frequency moves downstream with the wave velocity c_r ; therefore,

$$\frac{dx}{dt} = c_r$$

Then

$$A = e^{\int_{x_0}^x \frac{\alpha c_1}{c_r} dx} \quad (2)$$

To express the ratio A as a function of the boundary-layer Reynolds number, note that on a plate

$$R_{\delta*} = C_1 \sqrt{\frac{U_\delta x}{\nu}} \quad (3)$$

or

$$2R_{\delta*} dR_{\delta*} = C_1^2 \frac{U_\delta}{\nu} dx \quad (4)$$

When equation (4) is used, equation (2) becomes, for a flat plate,

$$A_p = e^{\frac{2}{C_1^2} \int_{R_{\delta*,0}}^{R_{\delta*}} \alpha \delta^* \frac{c_1}{c_r} dR_{\delta*}} \quad (5)$$

To obtain the corresponding expression for a cone, replace expression (3) by the relation between $R_{\delta*}$ and $\sqrt{\frac{U_\delta x}{\nu}}$ on a cone (ref. 2); namely,

$$R_{\delta*} = \frac{C_1}{\sqrt{3}} \sqrt{\frac{U_{\delta} x}{\nu}} \quad (6)$$

or

$$6R_{\delta*} dR_{\delta*} = C_1^2 \frac{U_{\delta}}{\nu} dx \quad (7)$$

When equation (7) is used, equation (2) becomes

$$A_c = e^{\frac{6}{C_1^2} \int_{R_{\delta*,0}}^{R_{\delta*}} \alpha_{\delta*} \frac{C_1}{c_r} dR_{\delta*}} \quad (8)$$

If both the initial and the final values of $R_{\delta*}$ are the same on plate and cone and if the disturbance travels along the same path in the same $\alpha_{\delta*}, R_{\delta*}$ plane for both cone and plate, the exponents in equations (5) and (8) differ by the factor 3. In this case it follows from equations (5) and (8) that

$$A_c = A_p^3 \quad (9)$$

The relation (9) requires the same path in the $\alpha_{\delta*}, R_{\delta*}$ plane and, therefore, the disturbance must have the same frequency on cone and plate. The relation (9) results because the disturbance takes three times as long to pass from $R_{\delta*,0}$ to $R_{\delta*}$ on the cone as on the plate and thus has three times as long in which to amplify. Note that for the $\alpha_{\delta*}, R_{\delta*}$ diagram to be identical for cone and plate, it is sufficient that the Mach number and the temperature boundary conditions on cone and plate be identical.

Equation (9) can also be obtained from Schlichting's work (ref. 5). Although Schlichting made calculations for a single frequency at a time, the disturbance was assumed to be composed of many frequencies. The velocity c_r was therefore replaced by the group velocity $c_r + \alpha \left(\frac{\partial c_r}{\partial \alpha} \right)$; the disturbance under consideration now moves with the group velocity. That the result given by equation (9) is unchanged can be seen by replacing c_r by $c_r + \alpha \left(\frac{\partial c_r}{\partial \alpha} \right)$ in equations (2), (5), and (8).

Equation (9) gives the relation between the amplification on the cone and that on the plate for a disturbance of the same time frequency that has reached the same value of R_{δ^*} on cone and plate. For example, if a disturbance of a single frequency has arrived at a value of R_{δ^*} , say $R_{\delta^*,1}$ (fig. 1), by moving along the path labeled (1) and has amplified A times on a plate, it will have amplified A^3 times on the cone along the same path (1). Disturbances of all frequencies can exist but only some cross the lower part of the neutral stability boundary (curve $c_1 = 0$, fig. 1) and pass into the region of amplification as they move downstream. The various disturbances of fixed frequency, for example, those along paths (2) and (3) in figure 1, cross the lower branch of the neutral curve at different values of R_{δ^*} , and by the time they have arrived at the same value of R_{δ^*} they have grown by different amounts. At each value of R_{δ^*} one frequency has grown more than any other. If the frequency for maximum amplification and the maximum amplification are known for each value of R_{δ^*} on the flat plate, then, because the $\alpha\delta^*, R_{\delta^*}$ diagram is the same for cone and plate, the maximum amplification to the same value of R_{δ^*} on a cone occurs for the same frequency and can be found from equation (9). That is, equation (9) becomes

$$A_{c,max} = (A_{p,max})^3 \quad (10)$$

Thus if, for example, the relation between $A_{p,max}$ and R_{δ^*} is given in the form of a curve of $A_{p,max}$ against R_{δ^*} , the same curve will be the relation between $A_{c,max}$ and R_{δ^*} , if the ordinate $A_{p,max}$ is replaced by $(A_{c,max})^{1/3}$.

Now let the relation between $A_{p,max}$ and R_{δ^*} be

$$\log_e A_{p,max} = F\left(R_{\delta^*,p}, M, \frac{T_w}{T_\delta}\right) \quad (11)$$

where $F\left(R_{\delta^*,p,min}, M, \frac{T_w}{T_\delta}\right) = 0$ because $A_{p,max} = 1$ at $R_{\delta^*,p} = R_{\delta^*,p,min}$.

The comparison between cone and plate is made for equal values of M and $\frac{T_w}{T_\delta}$. Then, because of equation (10) and the preceding discussion, the relation for the cone is

$$\log_e (A_{c,max})^{1/3} = F(R_{\delta^*,c}) \quad (12)$$

where M and T_w/T_δ are not written in equation (12) because $M_c = M_p$ and $\left(\frac{T_w}{T_\delta}\right)_c = \left(\frac{T_w}{T_\delta}\right)_p$. Equation (12) can also be written as

$$\log_e A_{c,max} = 3F(R_{\delta^*,c}) \quad (13)$$

The expression for the shear caused by the oscillations in the laminar boundary layer is taken as

$$\tau_{turb} = -\rho \overline{uv}$$

for compressible as well as for incompressible flow. (See ref. 6.)

The assumption is now made that the maximum value of the ratio of the disturbance shear

$$\tau_{turb,max} = -\rho \overline{uv}$$

to the laminar shear

$$\tau_{lam} = \mu \frac{dU}{dy}$$

in a boundary-layer cross section is a measure of the closeness to transition of the boundary layer at that cross section. The ratio is given by the expression

$$\frac{\tau_{turb,max}}{\tau_{lam}} = - \frac{2}{C_{f,lam}} \left\{ kb \left[\left(\frac{u'}{U_\delta} \right)_0 \right]^2 A^2 \right\}_{max} \quad (14)$$

where $k = \frac{\overline{uv}}{u'v'}$ is the correlation coefficient and b is the ratio of $\frac{v'}{U_\delta}$ to $\frac{u'}{U_\delta}$. The expression (14) is taken from reference 7, where it is derived and discussed.

For the plate, equation (14) becomes

$$\left(\frac{\tau_{\text{turb,max}}}{\tau_{\text{lam}}} \right)_p = - \frac{2}{C_{f,\text{lam},p}} \left\{ kb \left[\left(\frac{u'}{U_\delta} \right)_0 \right]^2 A^2 \right\}_{p,\text{max}} \quad (15)$$

For the cone, equation (14) becomes

$$\left(\frac{\tau_{\text{turb,max}}}{\tau_{\text{lam}}} \right)_c = - \frac{2}{C_{f,\text{lam},c}} \left\{ kb \left[\left(\frac{u'}{U_\delta} \right)_0 \right]^2 A^2 \right\}_{c,\text{max}} \quad (16)$$

To find the relation between cone and plate Reynolds numbers for equal closeness to transition, set the ratio $\left(\frac{\tau_{\text{turb,max}}}{\tau_{\text{lam}}} \right)_p$ equal to the ratio $\left(\frac{\tau_{\text{turb,max}}}{\tau_{\text{lam}}} \right)_c$. When $\left(\frac{\tau_{\text{turb,max}}}{\tau_{\text{lam}}} \right)_c$ is made equal to $\left(\frac{\tau_{\text{turb,max}}}{\tau_{\text{lam}}} \right)_p$ and equation (16) is divided by equation (15), the result is

$$\frac{\left[\left(\frac{u'}{U_\delta} \right)_{0,c} \right]^2 (A_{c,\text{max}})^2 C_{f,\text{lam},p}}{\left[\left(\frac{u'}{U_\delta} \right)_{0,p} \right]^2 (A_{p,\text{max}})^2 C_{f,\text{lam},c}} = 1 \quad (17)$$

where the assumption has been made that the magnitudes of the quantities k and b in equation (16) are about the same as their values in equation (15). This assumption is supported by the discussion in reference 7, which points out that the quantities k and b probably vary very slowly with Reynolds number.

The discussion in reference 7 cites Schlichting's calculations (ref. 8) for two points on the neutral curve ($c_1 = 0$) in the $\alpha\delta^*, R_{\delta^*}$ diagram (fig. 1). One point was at $R_{\delta^*} = 893$ on the lower branch of the neutral curve, the other at $R_{\delta^*} = 2070$ on the upper branch. Although k varied across the boundary layer, both its maximum value and the position for the maximum were almost the same for both points. Consequently, the available data do not contradict the assumption that k and b can be taken as approximately independent of the Reynolds number.

Equation (17) can also be written as

$$2 \log_e \frac{A_{c,max}}{A_{p,max}} + 2 \log_e \frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}} + \log_e \frac{C_{f,lam,p}}{C_{f,lam,c}} = 0 \quad (18)$$

Because the boundary-layer velocity profile on the plate and the cone is the same, it follows that

$$\frac{C_{f,lam,p}}{C_{f,lam,c}} = \frac{R_{\delta*,c}}{R_{\delta*,p}} \quad (19)$$

When equations (11), (13), and (19) are substituted into equation (18), the result is the general approximate relation between $R_{\delta*,c}$ and $R_{\delta*,p}$

$$3F(R_{\delta*,c}) - F(R_{\delta*,p}) + \log_e \frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}} + \frac{1}{2} \log_e \frac{R_{\delta*,c}}{R_{\delta*,p}} = 0 \quad (20)$$

No specific relation between $R_{\delta*,c}$ and $R_{\delta*,p}$ can be obtained from equation (20) unless the form of the function F is known. At present the only data available that can be used to obtain this function are those worked out by Schlichting for the incompressible flow over a flat plate (ref. 5). From Schlichting's results it is found (see fig. 2) that a good approximation for the function F is

$$F(R_{\delta*}) = S \left[R_{\delta*}^2 - (R_{\delta*,min})^2 \right] \quad (21)$$

where S is equal to 0.186×10^{-5} for Schlichting's data (fig. 2).

If relation (21) is used in equation (20) the relation between $R_{\delta*,c}$ and $R_{\delta*,p}$ becomes

$$3S \left[(R_{\delta*,c})^2 - (R_{\delta*,c,min})^2 \right] - S \left[(R_{\delta*,p})^2 - (R_{\delta*,p,min})^2 \right] + \log_e \frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}} + \frac{1}{2} \log_e \frac{R_{\delta*,c}}{R_{\delta*,p}} = 0 \quad (22)$$

Because the $\alpha\delta^*, R_{\delta^*}$ diagram is the same for cone and plate, it follows that $R_{\delta^*,p,min} = R_{\delta^*,c,min}$. Equation (22) can then be written as

$$3S(R_{\delta^*,c})^2 - S(R_{\delta^*,p})^2 - 2S(R_{\delta^*,min})^2 + \log_e \frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}} + \frac{1}{2} \log_e \frac{R_{\delta^*,c}}{R_{\delta^*,p}} = 0$$

or, after a slight rearrangement, as

$$\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 + \frac{1}{12S(R_{\delta^*,min})^2} \left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2 \log_e \left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 = \frac{1}{3} + \frac{2}{3} \left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2 - \frac{1}{3S(R_{\delta^*,min})^2} \left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2 \log_e \frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}} \quad \left(\begin{matrix} R_{\delta^*,c} \geq R_{\delta^*,min} \\ R_{\delta^*,p} \geq R_{\delta^*,min} \end{matrix}\right) \quad (23)$$

If the amplitude of the frequency that has had the maximum growth to $R_{\delta^*,c}$ is the same as the amplitude of the frequency that has had the maximum growth to $R_{\delta^*,p}$, when each frequency first crosses the lower branch of the neutral curve, then

$$\left(\frac{u'}{U_\delta}\right)_{0,c} = \left(\frac{u'}{U_\delta}\right)_{0,p}$$

Equation (23) then becomes

$$\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 + \frac{1}{12S(R_{\delta^*,min})^2} \left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2 \log_e \left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 = \frac{1}{3} + \frac{2}{3} \left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2 \quad (24)$$

Equation (23) is a relation between $\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2$ and $\left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2$ with the parameters $S(R_{\delta^*,min})^2$ and $\frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}}$. Equation (24) has the single parameter $S(R_{\delta^*,min})^2$.

If, in the absence of other data, the value of the parameter $S(R_{\delta^*,\min})^2$ is calculated from Schlichting's results (ref. 5), equation (24) can be solved for $\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2$ as a function of $\left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}}\right)^2$.

A convenient approach is to write equation (24) as

$$\left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}}\right)^2 = \frac{1 - 3\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2}{\left[\frac{1}{4S(R_{\delta^*,\min})^2} \log_e \left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 - 2\right]} \quad (25)$$

and to calculate $\left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}}\right)^2$ for various values of $\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2$. The result is shown in figure 3. This result can be represented by the simple form

$$\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 = \frac{1}{3} + \frac{2}{3}\left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}}\right)^2 \quad (26)$$

without introducing much more of an approximation. (See fig. 3.) Equation (26) can also be written as

$$3\left(R_{\delta^*,c}\right)^2 - \left(R_{\delta^*,p}\right)^2 = 2\left(R_{\delta^*,\min}\right)^2 \quad (27)$$

In order that equation (26) (or (27)) be a good approximation to equation (24) it follows from equation (25) that it is sufficient that

$$\frac{1}{4S(R_{\delta^*,\min})^2} \log_e \left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 \ll 2 \quad \left[\frac{1}{3} \leq \left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2 \leq 1 \right]$$

When this inequality is satisfied, equation (24) is essentially independent of the value of $S(R_{\delta^*,\min})^2$.

The relations that have been obtained involve boundary-layer Reynolds numbers. It is, however, often more convenient to consider Reynolds numbers based on the distance to the leading edge or to the apex rather than

on the boundary-layer thickness. In order to obtain relations that involve the Reynolds number R_x rather than $R_{\delta*}$, equation (22) is used, together with the relations (3), (6), and the relation

$$R_{x,c,min} = 3R_{x,p,min} \quad (28)$$

Equation (28) follows from $R_{\delta*,c,min} = R_{\delta*,p,min}$ and equations (3) and (6). Equation (22) can then be written as

$$3S \left(\frac{C_1^2}{3} R_{x,c} - \frac{C_1^2}{3} 3R_{x,p,min} \right) - S \left(C_1^2 R_{x,p} - C_1^2 R_{x,p,min} \right) + \\ \log_e \frac{\left(\frac{u'}{U_\delta} \right)_{0,c}}{\left(\frac{u'}{U_\delta} \right)_{0,p}} + \frac{1}{2} \log_e \frac{\sqrt{R_{x,c}}}{\sqrt{3} \sqrt{R_{x,p}}} = 0$$

or, after rearranging the terms, as

$$\left(\frac{R_{x,c}}{R_{x,p}} \right) + \frac{1}{4SC_1^2 R_{x,p,min}} \left(\frac{R_{x,p,min}}{R_{x,p}} \right) \log_e \frac{R_{x,c}}{3R_{x,p}} = 1 + 2 \left(\frac{R_{x,p,min}}{R_{x,p}} \right) - \\ \frac{1}{SC_1^2 R_{x,p,min}} \left(\frac{R_{x,p,min}}{R_{x,p}} \right) \log_e \frac{\left(\frac{u'}{U_\delta} \right)_{0,c}}{\left(\frac{u'}{U_\delta} \right)_{0,p}} \quad \left(\begin{array}{l} R_{x,c} \geq R_{x,c,min} \\ R_{x,p} \geq R_{x,p,min} \end{array} \right) \quad (29)$$

If the amplitude of the frequency that has had the maximum growth to $R_{x,c}$ is the same as the amplitude of the frequency that has had the maximum growth to $R_{x,p}$, when each frequency first crosses the lower branch of the neutral curve, then

$$\left(\frac{u'}{U_\delta} \right)_{0,c} = \left(\frac{u'}{U_\delta} \right)_{0,p}$$

and equation (29) becomes

$$\left(\frac{R_{x,c}}{R_{x,p}}\right) + \frac{1}{4SC_1^2 R_{x,p,\min}} \left(\frac{R_{x,p,\min}}{R_{x,p}}\right) \log_e \frac{R_{x,c}}{3R_{x,p}} = 1 + 2 \left(\frac{R_{x,p,\min}}{R_{x,p}}\right) \quad (30)$$

Equation (29) is a relation between $\left(\frac{R_{x,c}}{R_{x,p}}\right)$ and $\left(\frac{R_{x,p,\min}}{R_{x,p}}\right)$ with the parameters $SC_1^2 R_{x,p,\min}$ and $\left(\frac{u'}{U_\delta}\right)_{0,c}$. Equation (30) has the single parameter $SC_1^2 R_{x,p,\min}$. It is noted that if the Reynolds number $R_{x,p}$ becomes so large that the ratio $\frac{R_{x,p,\min}}{R_{x,p}}$ is near zero, then equation (29) indicates that the ratio $\frac{R_{x,c}}{R_{x,p}}$ approaches unity for all finite values of $\log_e \frac{\left(\frac{u'}{U_\delta}\right)_{0,c}}{\left(\frac{u'}{U_\delta}\right)_{0,p}}$.

If, in the absence of other data, the value of the parameter $SC_1^2 R_{x,p,\min}$ is calculated from Schlichting's results (ref. 5), the equation (30) can be solved explicitly for $\frac{R_{x,c}}{R_{x,p}}$ as a function of $\frac{R_{x,p,\min}}{R_{x,p}}$. A convenient approach is to write equation (30) as

$$\frac{R_{x,p,\min}}{R_{x,p}} = \frac{1 - \frac{R_{x,c}}{R_{x,p}}}{\left[\frac{1}{4SC_1^2 R_{x,p,\min}} \log_e \frac{R_{x,c}}{3R_{x,p}} - 2 \right]} \quad (31)$$

and to calculate $\frac{R_{x,p,\min}}{R_{x,p}}$ for various values of $\frac{R_{x,c}}{R_{x,p}}$. The result is shown in figure 4. This result can be represented by the simple form

$$\left(\frac{R_{x,c}}{R_{x,p}}\right) = 1 + 2 \left(\frac{R_{x,p,\min}}{R_{x,p}}\right) \quad (32)$$

without introducing much more of an approximation. (See fig. 4.) Equation (32) can also be written as

$$R_{x,c} - R_{x,p} = 2R_{x,p,\min} \quad (33)$$

In order that equation (32) or (33) be a good approximation to equation (30), it follows from equation (31) that it is sufficient that

$$\frac{1}{4SC_1^2 R_{x,p,\min}} \log_e \frac{R_{x,c}}{3R_{x,p}} \ll 2 \quad \left(\frac{1}{3} \leq \frac{R_{x,c}}{3R_{x,p}} \leq 1 \right)$$

When this inequality is satisfied, equation (30) is essentially independent of the value of $SC_1^2 R_{x,p,\min}$.

The basis of the relations between the cone and plate Reynolds numbers is the assumption that the measure of the closeness to transition is the ratio of the shear caused by the oscillations in the laminar boundary layer to the viscous shear. It is interesting to note that if this basis is replaced by the requirement that the amplitude of the disturbances on the cone and plate must be equal for equal closeness to transition, it can be shown that the relations (23) and (24) are replaced by

$$\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}} \right)^2 = \frac{1}{3} + \frac{2}{3} \left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}} \right)^2 - \frac{1}{3S(R_{\delta^*,\min})^2} \left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}} \right)^2 \log_e \frac{\left(\frac{u'}{U_\delta} \right)_{0,c}}{\left(\frac{u'}{U_\delta} \right)_{0,p}} \quad (34)$$

and

$$\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}} \right)^2 = \frac{1}{3} + \frac{2}{3} \left(\frac{R_{\delta^*,\min}}{R_{\delta^*,p}} \right)^2 \quad (35)$$

The relations (29) and (30) are replaced by

$$\left(\frac{R_{x,c}}{R_{x,p}} \right) = 1 + 2 \left(\frac{R_{x,p,\min}}{R_{x,p}} \right) - \frac{1}{SC_1^2 R_{x,p,\min}} \left(\frac{R_{x,p,\min}}{R_{x,p}} \right) \log_e \frac{\left(\frac{u'}{U_\delta} \right)_{0,c}}{\left(\frac{u'}{U_\delta} \right)_{0,p}} \quad (36)$$

and

$$\left(\frac{R_{x,c}}{R_{x,p}}\right) = 1 + 2\left(\frac{R_{x,p,min}}{R_{x,p}}\right) \quad (37)$$

Equation (35) is noted to be the same as equation (26) and equation (37), the same as equation (32). Therefore, if Schlichting's data are used, the same result is obtained whether the ratio of shears or the amplitude is used as a criterion.

DISCUSSION

A general relation (eq. (20)) between the Reynolds number on a cone and that on a plate for equal closeness to transition has been obtained. The relation is based on the linear stability theory and on the assumption that the ratio of the laminar shear to the shear caused by the velocity fluctuations is an indication of the closeness to transition. In order that equation (20) be valid it is also necessary that equation (1) be correct. The use of equation (1) for a cone is based on the assumption that the ratio of the boundary-layer thickness to the cone radius is so small that the linearized equations of continuity, motion, and energy are the same for the cone as for the plate. Consequently the values of α , c_r , and c_i at a point on a cone are then the same as at a point on a plate when the pertinent properties of the boundary layer at the two points are identical. It is also assumed that the amplification or decay of a disturbance as it moves downstream can be calculated in the same way as on a plate; that is, that although the effect of the downstream change in surface area of the cone becomes more important as the ratio of the boundary-layer thickness to the cone radius increases (and also as the ratio of the cone radius at transition to the cone radius at the beginning of amplification increases), the effect is not important enough to change the main conclusion; namely, that when $R_{x,p}$ is small, the ratio $\frac{R_{x,c}}{R_{x,p}}$ is near 3, but that when $R_{x,p}$ is very large the ratio $\frac{R_{x,c}}{R_{x,p}}$ is near unity.

An attempt was made to retain the radius of the cone in the linearized partial differential equations of motion and continuity, but when the radius of the cone was retained, the variables in the partial differential equations could not be separated. This means that disturbances of the type upon which the analysis is based, namely of the type $\phi(y)e^{i\alpha(x-ct)}$, are not, in general, possible on a cone. For vanishingly small values of $\frac{\delta}{r}$, however, the radius disappears from the equations and the equations become those for a plate; disturbances of the type $\phi(y)e^{i\alpha(x-ct)}$ are then allowable.

An analysis based on the linear stability theory cannot, of course, provide a relation between transition Reynolds numbers on cones and those on plates because, when transition occurs, the disturbances are too large for their behavior to be described by the linear stability theory. The relations for equal closeness to transition can thus be useful in a comparison of cone and plate transition Reynolds numbers only if the difference between the Reynolds numbers at which the relations for equal shear ratio become inapplicable and the transition Reynolds numbers is not appreciable. There is no "a priori" method for determining whether this difference is appreciable, because there is no sharp dividing line between those oscillations with magnitudes small enough for their behavior to be described by linear theory and those with magnitudes too large to be so described. Moreover, although the use of the shear ratio as an indication of the closeness to transition seems reasonable, because the shear caused by the velocity fluctuations varies from a negligible amount in purely laminar flow to essentially the entire amount in turbulent flow, it does remain an assumption.

The specific relations, equations (23), (24), (29), and (30), are obtained from equation (20) by the use of the approximation given by equation (21). Although equation (21) represents Schlichting's data almost exactly, it is possible that, for other Mach numbers and surface temperature ratios, the linear approximation (eq. (21)) will not be so exact. In this case the linear approximation can still be used if the resulting decrease in accuracy is not important; or else a parabolic or higher degree approximation can be used. At present no data on the behavior of amplified oscillations in the compressible laminar boundary layer are available; therefore, Schlichting's data are used to find the value of the parameter S in equation (21). Because Schlichting's data on amplification are used, the minimum critical Reynolds number is taken as 575, the value calculated by Schlichting.

In order to determine whether equations (29), (30), and (33) are useful in the comparison of cone and plate transition Reynolds numbers, the data in references 9 and 10 are used. The conclusion from reference 9 is that the ratio $\left(\frac{R_{x,c}}{R_{x,p}}\right)_T$ is about 3. If the factor $2 \frac{R_{x,p,min}}{R_{x,p}}$ in equation (29) is assumed to be negligible compared with unity and if the value 0.555×10^{-5} is used for the term SC_1^2 , the result is that

$\left(\frac{u'}{U_\infty}\right)_{0,p} \approx 2350 \left(\frac{u'}{U_\infty}\right)_{0,c}$ for $R_{x,p} = 7 \times 10^5$; that is, the initial disturbances on the plates in the data cited in reference 9 would have to be very much larger than those on the cones in order that the relation (29) predict the behavior shown in reference 9. Reference 9 discusses this possibility.

On the other hand, the data of reference 10 behave in the manner indicated by equation (33). It seems clear that a larger amount of

comparable data is needed for cones and plates before a definite conclusion can be drawn concerning the usefulness of the approximate relations derived in the present analysis. At supersonic speeds the leading-edge radius is noted to have an effect on the transition Reynolds number on plates and cones; and this effect may be large enough to affect the correlation between plate and cone transition Reynolds numbers.

In closing, it is remarked that although the present state of development of the theory does not permit the calculation of the ratio $\left(\frac{R_{x,c}}{R_{x,p}}\right)_T$, the indication is that the ratio $\left(\frac{R_{x,c}}{R_{x,p}}\right)_T$ will be near 3 only when transition occurs near the minimum critical Reynolds number. When transition occurs at a much larger Reynolds number, the indication is that the ratio $\left(\frac{R_{x,c}}{R_{x,p}}\right)_T$ will be near unity. Because these remarks apply only when transition is caused by the amplification of very small disturbances, they are not applicable when local surface imperfections or distributed roughness cause transition by generating large disturbances.

CONCLUDING REMARKS

The linear theory of boundary-layer stability is used to derive an approximate relation between the Reynolds number on a cone and the Reynolds number on a flat plate for equal closeness to transition; the assumption is made that the ratio of the laminar shear to the shear caused by the fluctuations in the laminar boundary layer is an indication of the closeness to transition. The fluctuations on plate and cone are assumed to be of the same type and to be periodic in the direction of flow. By use of Schlichting's calculated amplification ratios for incompressible flow, the approximate relation is made specific. This specific relation is roughly that, at equal ratio of oscillation shear to laminar shear, the cone Reynolds number based on the distance to the apex exceeds the plate Reynolds number based on the distance to the leading edge by twice the minimum critical Reynolds number on the plate. This relation requires that the amplitude of the disturbance be equal on cone and plate where amplification begins. The frequency on the cone is the frequency that results in maximum amplification at a Reynolds number; the frequency on the plate is the frequency that results in the maximum amplification at the corresponding Reynolds number on the plate and is in general not the same as the frequency on the cone.

Although an exact analysis of the transition problem is not given nor even an exact analysis of the stability of the laminar boundary layer on a cone, the indication is that the ratio of the cone Reynolds number for transition (based on the distance to the cone apex) to the

plate Reynolds number for transition (based on distance to the leading edge) is not in general equal to 3, as has been suggested by other investigators. The analysis indicates that the ratio varies from 3 when transition occurs at the minimum critical Reynolds number to unity when transition occurs at a large multiple of the critical Reynolds number. An examination of two sets of data does not lead to a definite conclusion concerning the validity of the results obtained.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., June 7, 1957.

REFERENCES

1. Lin, C. C.: The Theory of Hydrodynamic Stability. Cambridge Univ. Press, 1955.
2. Hantsche and Wendt: The Laminar Boundary Layer on a Circular Cone at Zero Incidence in a Supersonic Stream. Reps. and Translations No. 276, British M.A.P. Völkenrode, Aug. 1946.
3. Van Driest, E. R.: Turbulent Boundary Layer on a Cone In a Supersonic Flow at Zero Angle of Attack. Jour. Aero. Sci., vol. 19, no. 1, Jan. 1952, pp. 55-57, 72.
4. Battin, R. H., and Lin, C. C.: On the Stability of the Boundary Layer Over a Cone. Jour. Aero. Sci., vol. 17, no. 7, July 1950, pp. 453-454.
5. Schlichting, H.: On the Production of Turbulence in Flow Along a Plate. Translation No. 6, Council for Sci. and Ind. Res., Div. Aero., Commonwealth of Australia, July 1944.
6. Shen, S. F.: Studies of Von Kármán's Similarity Theory and Its Extension to Compressible Flows - Investigation of Turbulent Boundary Layer Over a Flat Plate in Compressible Flow by the Similarity Theory. NACA TN 2543, 1951.
7. Liepmann, H. W.: Investigation of Boundary Layer Transition on Concave Walls. NACA WR W-87, 1945. (Formerly NACA ACR 4J28.)
8. Schlichting, H.: Amplitude Distribution and Energy Balance of Small Disturbances in Plate Flow. NACA TM 1265, 1950.
9. Probstein, Ronald F., and Lin, C. C.: A Study of the Transition to Turbulence of the Laminar Boundary Layer at Supersonic Speeds. Preprint No. 596, S.M.F. Fund Preprint, Inst. Aero. Sci., Inc., Jan. 1956.
10. Laufer, John, and Marte, Jack E.: Results and a Critical Discussion of Transition-Reynolds-Number Measurements on Insulated Cones and Flat Plates in Supersonic Wind Tunnels. Rep. No. 20-96, (Contract No. DA-04-495-Ord 18), Jet Propulsion Lab., C.I.T., Nov. 30, 1955.

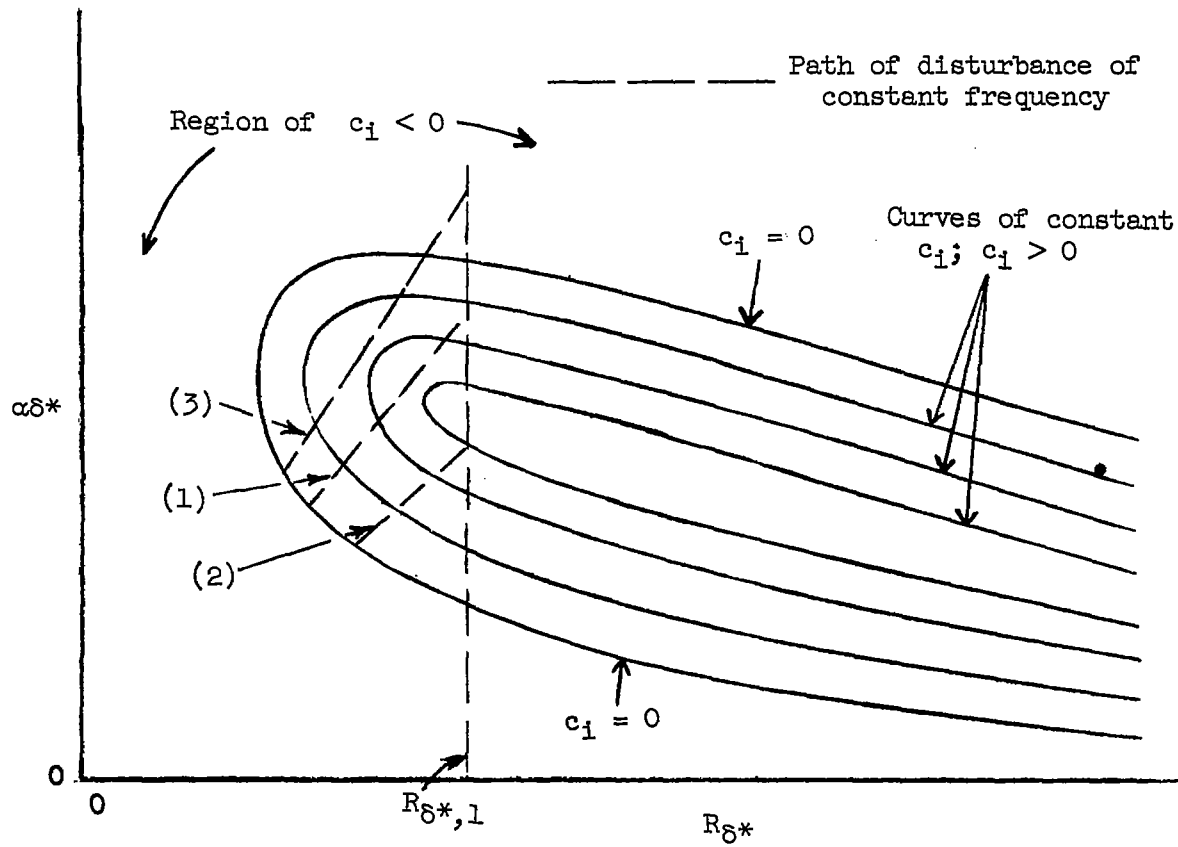


Figure 1.- The amplification c_1 as a function of the Reynolds number R_{δ^*} and the wave number $\alpha\delta^*$.

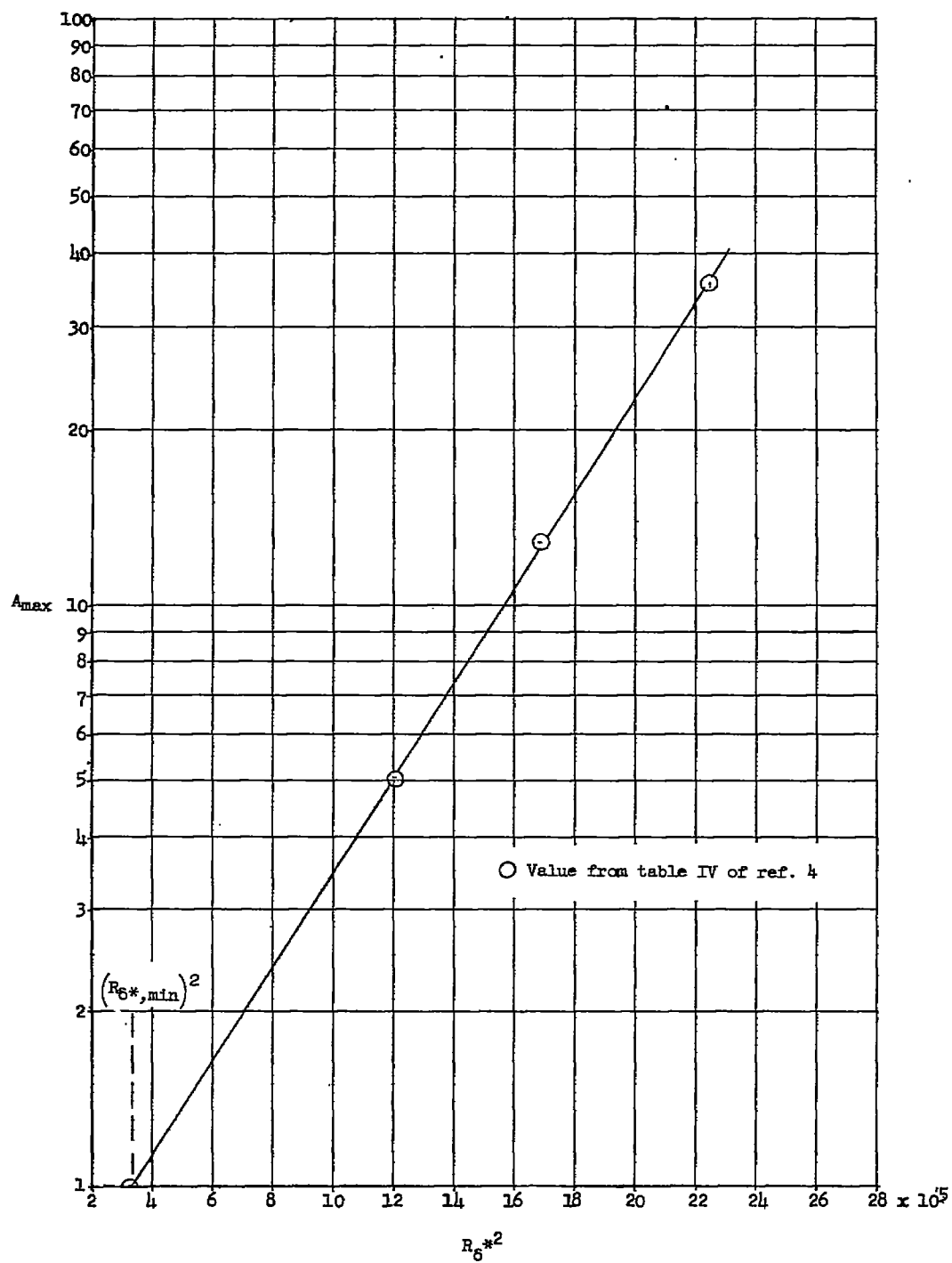


Figure 2.- Maximum amplification as a function of R_δ^{*2} .

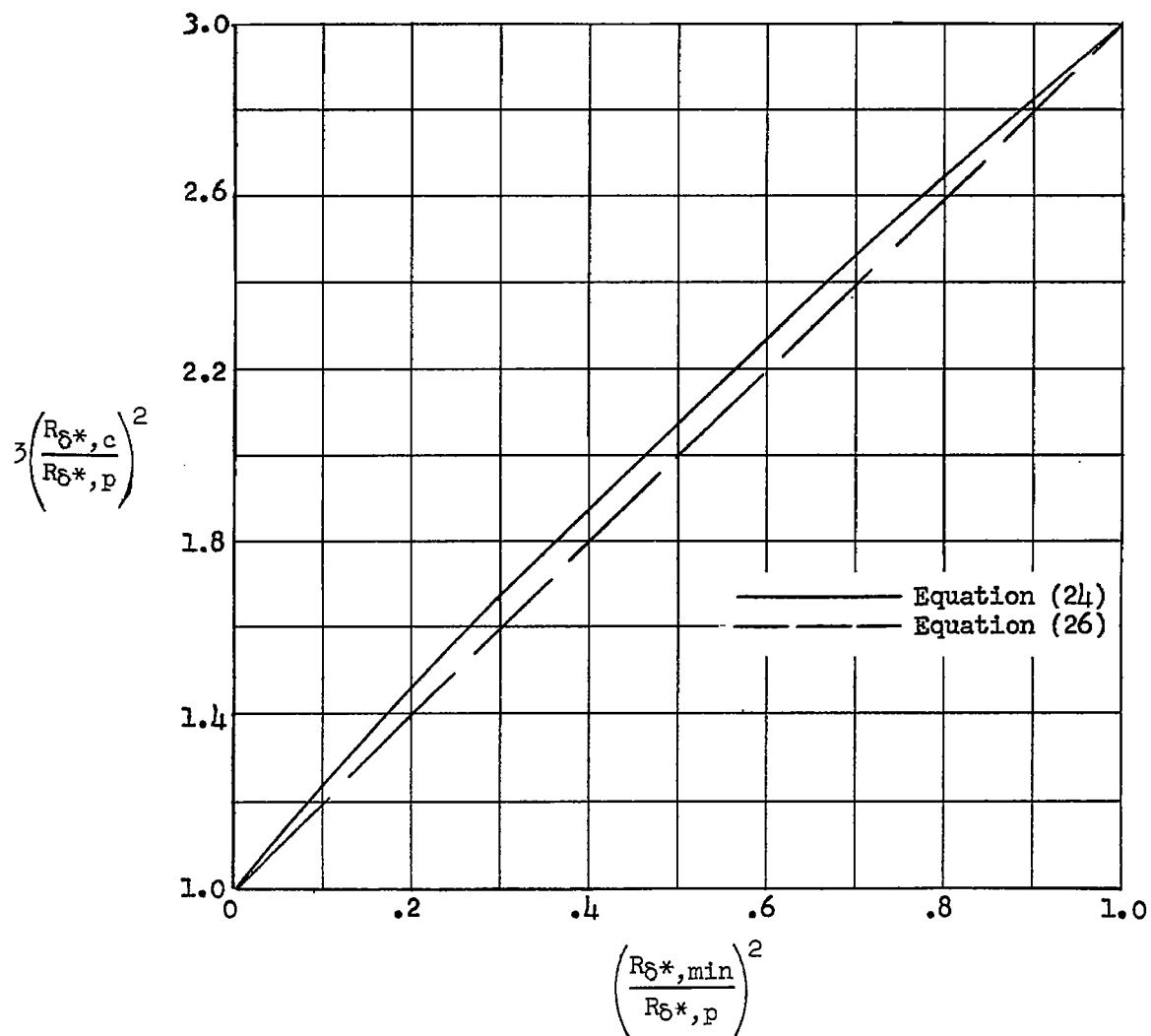


Figure 3.- Variation of $\left(\frac{R_{\delta^*,c}}{R_{\delta^*,p}}\right)^2$ with $\left(\frac{R_{\delta^*,min}}{R_{\delta^*,p}}\right)^2$ for equal ratios of oscillation shear to laminar shear on cone and plate.

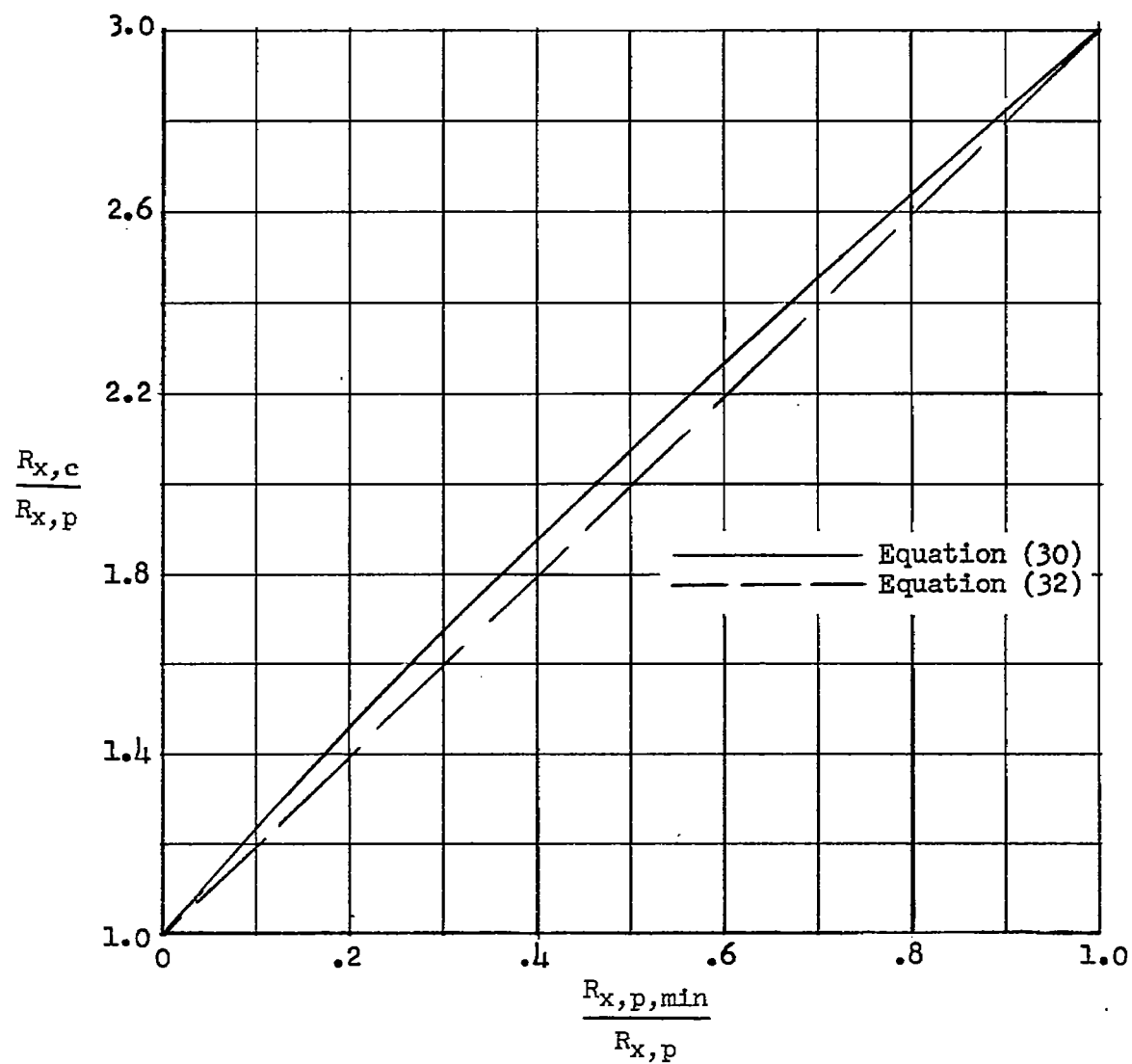


Figure 4.- Variation of $\frac{R_{x,c}}{R_{x,p}}$ with $\frac{R_{x,p,min}}{R_{x,p}}$ for equal ratios of oscillation shear to laminar shear on cone and plate.